Math 246A Lecture 19 Notes

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November 7, 2018

1 General Cauchy's Theorem and Finitely Connected Domains

1.1 General Cauchy's theorem for 1-forms

Let's finish the proof from last time.

Theorem 1.1. *Let Ω be a domain and $\gamma \subseteq \Omega$ be a piecewise C^1 cycle homologous to 0. For a closed C^2 1-form $P\,dx + Q\,dy$,*

$$\int_\gamma P\,dx + Q\,dy = 0.$$

Proof. Last time we found a curve $\sigma \subseteq \Omega$ made of horizontal and vertical segments such that

$$\int_\sigma P\,dx + Q\,dy = \int_\gamma P\,dx + Q\,dy$$

for all closed $P\,dx + Q\,dy$. Extend all segments in σ to horizontal or vertical lines. These bound some bounded rectangles $R_1, \dots, R_n$ and some unbounded rectangles $R_1', \dots, R_m'$. Pick an $a_j \in R_j$ for each j. Let

$$\tilde{\sigma} = \sum_{j=1}^n n(\sigma, a_j)\partial R_j.$$

There are two steps in the proof:

1. $\tilde{\sigma} = \sigma$: Suppose $n(\sigma, a_j) \neq 0$. Assume that $\tilde{\sigma} - \sigma$ contains $m\sigma_{i,j}$, where m in $\mathbb{Z}$, and $\sigma_{i,j} \subseteq \partial R_i \cap \partial R_j'$. Let $\tau = \tilde{\sigma} - \sigma - m\partial R_i$. Note that $n(\tau, a_j) = n(\tau, a_i) = 0$ because these points are connected. This is impossible, so $m = 0$.

2. $n(\sigma, a_j) \neq 0 \implies \overline{R_j} \subseteq \Omega$: $n(\sigma, a_j) \neq 0$ implies that $a_j \in \Omega$, so we get $R_j^o \subseteq \Omega$. If U is the component of $\mathbb{C} \setminus \mathbb{C}$ with $R_j^o \subseteq U$, then $\overline{R_j} = (\overline{R_j} \cap \sigma) \cup \overline{R_j} \cap U)$. So $\overline{R_j} \subseteq \Omega$.

These two imply that

$$\begin{aligned}\int_\gamma P\,dx + Qdy &= \int_\sigma P\,dx + Qdy \\ &= \int_{\tilde{\sigma}} P\,dx + Qdy \\ &= \sum_{j=1}^{n} n(\sigma, a_j) \int_{\partial R_j} P\,dx + Qdy \\ &= 0. \end{aligned}$$

□

1.2 Finitely connected domains

Let $\Omega \subseteq \mathbb{C}^*$ be a domain such that $\mathbb{C}^* \setminus \Omega = K_0 \cup \cdots \cup K_N$ with K_j connected, $K_j \neq \varnothing$, K_j closed, and $K_j \cap K_k \neq \varnothing$ for $j \neq k$. Without loss of generality, $\infty \in K_0$, so we can say $\Omega \subseteq \mathbb{C}$.

A simply connected domain is when $N = 0$, so without loss of generalty, $N \geq 1$. Now let $0 < \delta < \inf_{j \neq k} \operatorname{dist}(K_j, K_k)$, and let $(S_p)_{p \in \mathbb{Z}}$ be squares of side length $\delta/\sqrt{2}$ with sides parallel to the axes, which tile the plane. For $j = 1, \ldots, N$, let

$$\gamma_j = \sum_{S_p \cap K_j \neq \varnothing} \partial S_p.$$

Then $\gamma_j \subseteq \Omega$, and $n(\gamma_j, a_k) = \delta_{j,k}$, where $a_k \in K_k$. The set $\{\gamma_1, \ldots, \gamma_N\}$ is a homology basis for Ω.

Definition 1.1. A homology basis $\{\gamma_1, \ldots, \gamma_N\}$ is a collection of closed curves such that for any cycle γ, $\gamma - c_i\gamma_1 - \cdots - c_N\gamma_N \sim 0$ for $c_j \in \mathbb{Z}$.

Theorem 1.2. *Let $f \in H(\Omega)$ with $f(z) \neq 0$ for all $z \in |omega$. Then there exists unique $n_1, n_2, \ldots, n_N \in \mathbb{Z}$ and a unique $g \in H(\Omega)$ such that*

$$f(z) = \prod_{j=1}^{\infty} (z - a_j)^{n_j} e^{g(z)}$$

Proof. By the argument principle

$$\frac{1}{2\pi i} \int_{\gamma_j} \frac{f'(z)}{f(z)}\, dz = n(f(\gamma_j), 0).$$

Let

$$h(z) = \frac{f(z)}{\prod_{j=1}^{N} (z - a_j)^n}.$$

Then

$$\frac{1}{2\pi i}\int_{\gamma_j}\frac{h'(z)}{h(z)}\,dz = 0,$$

so $h'/h = g'$ for some $g \in H(\Omega)$. So $\log(h) = g$, and we get the result. $\square$

Corollary 1.1. *Let $f \in H(\Omega)$ with $f(z) \neq 0$ for all $z \in \Omega$. Let $k \in \mathbb{N}$. Then there exists $g \in H(\Omega)$ such that $f = g^k$ if and only if*

$$\frac{1}{2\pi i}\int_{\gamma_j}\frac{f'(z)}{f(z)}\,dz \in k\mathbb{Z}$$

for all $j = 1, \dots, N$.

Theorem 1.3. *Assume Ω is finitely connected with $K_1, \dots, K_N$ and $\gamma_1, \dots, \gamma_N$ as before. Let $u : \Omega \to \mathbb{R}$ be C^2 and harmonic. Then there exist unique $\alpha_1, \alpha_2, \dots, \alpha_N \in \mathbb{R}$ such that*

$$\sum_{j=1}^{N}\alpha_j \log|z - a_j| = \mathrm{Re}(f), \qquad \alpha_j = \frac{1}{2pi}\int_{\gamma_j} *du$$

for $f \in H(H)$. Moreover, f is unique up to the addition of a purely imaginary constant.